**Indian Statistical Institute, Bangalore**

M. Math.I Year, First Semester

Semestral Examination

Algebra -I

Time: 3 hours November 30, 2009 Instructor: N.S.N.Sastry

Maximum Marks 100

Answer all questions. Your arguments should be complete and to the point.

1. Let $R$ be a commutative ring with 1. Define a free module over $R$ of a given rank. Show that two free $R$ module $M$ and $N$ are isomorphic if, and only if, their ranks are the same. [4+8]

2. Define the tensor product of two modules over a commutative ring with 1. For positive integers $m$ and $n$, determine the structure of the tensor product of a cyclic group of order $m$ and a cyclic group of order $n$. Justify your answer. [4+8]

3. Using the universal property of the tensor product of two modules over a commutative ring $R$ with 1, show that, for $R$- modules $M, N$ and $L$, $(M \otimes N) \otimes L$ is isomorphic to $M \otimes (N \otimes L)$. [12]

4. Construct a non-abelian group of order $p^7, p$ a prime, whose centre is a cyclic group of order $p$. Justify your answer. [12]

5. Let $V$ be a vector space over a field $k$. Define the exterior algebra associated with $V$.

   If $\dim_k V = n < +\infty$, determine the dimension of the exterior algebra associated with $V$. [6+6]

6. Let $p$ be a prime number. Show that a Sylow $p$- subgroup of the symmetric group $S_{p^2}$ on $p^2$ elements is isomorphic to a wreath product of a cyclic group of order $p$ with itself. [12]

7. For each of the following properties below, give an example of a ring $R$ and an $R-$ module satisfying the property:

   (i) Noetherian but not Artinian,

   (ii) Artinian but not Noetherian ;

   (iii) Neither Noetherian nor Artinian ;

   (iv) Projective but not free.

   Justify your answer. [3+3+4+3]

8. Determine the number of group of the orders given below, up to isomorphism:

   (i) 35 (ii) 21 [6+9]

end